Rational second kind Chebyshev approximation for solving some physical problems on semi-infinite intervals

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ABSTRACT

In this paper, we introduce a new numerical technique to solve some physical problems on a semi-infinite interval. The approach is based on a rational second kind Chebyshev tau method. The operational matrices of derivative and product of rational second kind Chebyshev functions are presented and two nonlinear examples are solved. In the first example, the Volterra’s population growth model is formulated as a nonlinear differential equation, and in the second example, the Lane-Emden nonlinear differential equation is considered. Present method is utilized to reduce the solution of these physical problems to the solution of systems of algebraic equations. The method is easy to implement and yields very accurate results.

Keywords: Chebyshev polynomials of second kind; rational second kind Chebyshev functions; operational matrix of derivative; the product operational matrix; the Tau method.

INTRODUCTION

In the study of physical and engineering structures, ordinary differential equations over infinite intervals play an important role. Two important equations are: the Volterra’s population model and the Lane-Emden equation.

The Volterra’s model for population growth of a species within a closed system is given (Scudo, 1971) as

\[ k \frac{du}{dt} = u - u^2 - u \int_0^t u(x)dx, (0) = u_0, \]  

(1)

where \( u(t) \) is the scaled population of identical individuals at time \( t \), and \( k \) is a prescribed parameter. The nondimensional parameter \( k = c/(ab) \), where \( a > 0 \) is the birth rate coefficient, \( b > 0 \) is the crowding coefficient, and \( c > 0 \) is the toxicity coefficient. The coefficient \( c \) indicates the essential behavior of the
population evolution before its level falls to zero in the long run. Volterra
introduced this model for a population \( u(t) \) of identical individuals which
exhibits crowding and sensitivity to the amount of toxins produced. The
interested reader is referred to Scudo (1971) and TeBeest (1997) for detailed
discussions of the Volterra’s model. A considerable research work has been
invested recently by Scudo (1971), Small & Klamkin (1989), TeBeest (1997) and
Wazwaz (1999) among others to the development of efficient strategies to
determine numerical and analytic solutions of the population growth model in
Eq. (1). Although a closed form solution has been achieved by Small & Klamkin
(1989) and TeBeest (1997), it was formally shown that the closed form solution
cannot lead to an insight into the behavior of the population evolution. As a
result, a great deal of interest was directed towards the analysis of the rapid
population rise along the logistic curve followed by its decay to zero in the long
run. The nondimensional parameter \( k \) plays a great role in the behavior of \( u(t) \)
concerning the rapid rise to a certain amplitude followed by an exponential
decay to extinction.

Some approximate and numerical solutions for the Volterra’s population
model have been reported. In Small & Klamkin (1989), singular perturbation
methods were implemented to handle the two major cases that arise when \( k \) is
small and when \( k \) is large. More recently, three numerical algorithms were
applied independently in TeBeest (1997). The Euler and the modified Euler
method, the fourth order Runge-Kutta method, and a phase-plane analysis were
implemented. In TeBeest (1997), the numerical results were correlated to give an
insight of the problem and its solution without using singular perturbation
techniques. However, the performance of the traditional numerical techniques is
well known in that it provides grid points only, and in addition, it requires a
large amounts of calculations. In Wazwaz (1999), the series solution method and
the decomposition method are implemented independently to Eq. (1) and to a
related ordinary differential equation. Furthermore, the Padé approximations
are used in the analysis to capture the essential behavior of the population \( u(t) \)
of identical individuals.

The Lane-Emden equation of index \( m \) is a basic equation in the theory of
stellar structure (Shawagfeh, 2002). The equation describes the temperature
variation of a spherical gas cloud under the mutual attraction of its molecules
and subject to the laws of thermodynamics (Chandrasekhar, 1967; Shawagfeh,
2002). The Lane-Emden equation of index \( m \) is of the form

\[
xy''(x) + 2y'(x) + xy''(x) = 0, \quad x \geq 0
\]
which has been the object of much studies (Shawagfeh, 2002; Adomian, et al., 1995). The boundary conditions, which are of most interest (Davis, 1962), are the following:

\[ y(0) = 1, \ y'(0) = 0. \]  

It was shown physically that interesting values of \( m \) lie in the interval \([0, 5]\). In addition, exact solutions exist only for \( m = 0, 1 \) and 5. For other values of \( m \), series solutions are obtainable. Moreover Eq. (2) is linear for \( m = 0 \) and 1, and nonlinear otherwise. Recently, (He, 2003) proposed a variational approach to Eq. (2). Bender, et al. (1989) proposed a perturbation technique based on an artificial parameter \( \delta \); the method is often called \( \delta \)-method (Bender, et al., 1989; Andrianov & Awrejcewicz 2000). Moreover, Wazwaz (2001) applied the Adomian decomposition method to solve the Lane-Emden equation of index \( m \).

In this paper, we use rational second kind Chebyshev (RSC) functions for the solution of Volterra’s population model and Lane-Emden equation. The method consists of expanding the solutions of Eqs. (1) and (2) by RSC functions with unknown coefficients. The operational matrices of derivative and product of RSC functions are given. These matrices together with the tau method (Lanczos, 1987) are then utilized to evaluate the unknown coefficients and find approximate solutions in each case. Tau method is based on expanding the required approximate solution as the elements of a complete set of orthogonal functions. In tau method, unlike the Galerkin approximation, the expansion functions are not required to satisfy the boundary constraint individually (Canuto, et al., 1986; Gottlieb et al., 1984).

This paper is organized as follows: In Section 2, we describe the formulation of RSC functions required for our subsequent development. In Section 3, the Volterra’s population model is considered. This equation is first converted to an equivalent nonlinear ordinary differential equation, then the solution is approximated by a RSC tau method. As a result a set of nonlinear algebraic equations is formed, and a solution of the considered ODE is obtained. Section 4 summarizes the application of the RSC tau method to the solution of the Volterra’s model and the Lane-Emden equation.

**PROPERTIES OF RSC FUNCTIONS**

**RSC functions**

The second kind Chebyshev polynomials \( U_n(x), n = 0, 1, \ldots \), are orthogonal in the interval \([-1, 1]\) with respect to the weight function \( \sqrt{1 - x^2} \). It is known that \( U_n(x) \) satisfies the recurrence relation (Mason, 2003)
\[ U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x), \quad n = 2, 3, \ldots \]

\[ U_0(x) = 1, \quad U_1(x) = 2x. \]

The RSC functions are defined by

\[ R_n(x) = U_n\left(\frac{x-1}{x+1}\right), \]

thus, RCS functions must satisfy the following recurrence relation

\[ R_n(x) = 2\left(\frac{x-1}{x+1}\right)R_{n-1}(x) - R_{n-2}(x), \quad n \geq 2 \quad (4) \]

\[ R_0(x) = 1, \quad R_1(x) = 2\left(\frac{x-1}{x+1}\right). \quad (5) \]

RSC functions are orthogonal with respect to the weight function

\[ w(x) = \frac{4\sqrt{x}}{(x+1)^3} \] on the interval \([0, \infty)\) with the orthogonality property

\[ \int_0^\infty R_n(x)R_m(x)w(x)\,dx = \frac{\pi}{2} \delta_{nm}, \]

where \(\delta_{nm}\) is the Kronecker delta-function.

**Function approximation**

Since the set of RSC functions is orthogonal and complete, every function \(y \in L^2[0, \infty)\) can be expanded as

\[ y(x) = \sum_{i=0}^{\infty} c_i^R R_i(x), \quad (6) \]

where

\[ c_i^R = \frac{2}{\pi} \int_0^{\infty} (x)y(x)w(x)\,dx. \]

If the infinite series in Eq. (\ref{eq6}) is truncated, then it can be written as
\[ y(x) \approx \sum_{i=0}^{n-1} c_i^R R_i(x) = A^T R(x), \]

with

\[ A = \begin{bmatrix} c_0^R & c_1^R & \ldots & c_{n-1}^R \end{bmatrix}, \tag{7} \]

\[ R(x) = [R_0(x) \quad R_1(x) \quad \ldots \quad R_{n-1}(x)]^T. \]

**Operational matrix of derivative**

The derivative of the vector \( R(x) \) defined in Eq. (7) can be approximated by

\[ R'(x) \approx DR(x), \tag{8} \]

where \( D \) is the \( n \times n \) operational matrix for derivative. Differentiating Eqs. (5) and (4) we get

\[ R'_0(x) = 0, \quad R'_1(x) = \frac{5}{4} R_0(x) - R_1(x) + \frac{1}{4} R_2(x), \tag{9} \]

\[ R'_n(x) = (R_1(x)R_{n-1}(x))' - R'_{n-2}(x), \quad n \geq 2. \tag{10} \]

By using Eqs. (9) and (10) the matrix \( D \) can be calculated. The matrix \( D \) is a lower Hessenberg matrix and can be expressed as \( D = D_1 + D_2 \), where \( D_1 \) is a tridiagonal matrix which is obtained from

\[ D_1 = \text{diag} \left( \frac{-2 + 7i}{4}, -i, \frac{i}{4} \right), \quad i = 0, \ldots, n - 1 \]

and the \( d_{ij} \) elements of matrix \( D_2 \) are obtained from

\[ d_{ij} = \begin{cases} 0, & i \leq j + 1 \\ (-1)^{i+j+1}(2j), & i > +1. \end{cases} \]

For example, for \( n = 7 \) we have
\[ D = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{5}{4} & -1 & \frac{1}{4} & 0 & 0 & 0 & 0 \\
-2 & 3 & -2 & \frac{1}{2} & 0 & 0 & 0 \\
2 & -4 & \frac{19}{4} & -3 & \frac{3}{4} & 0 & 0 \\
-2 & 4 & -6 & \frac{13}{2} & -4 & 1 & 0 \\
2 & -4 & 6 & -8 & \frac{33}{4} & -5 & \frac{5}{4} \\
-2 & 4 & -6 & 8 & -10 & 10 & -6 
\end{bmatrix} \]

The product operational matrix

The following property of the product of two rational Chebyshev vectors will also be used.

\[ R(x)R^T(x)A = \tilde{A}R(x), \quad (11) \]

where \( \tilde{A} \) is an \( n \times n \) product operational matrix for vector \( A \). Using Eq. (11) and the orthogonal property, the elements \( \tilde{A}_{ij}, \ i = 0, \ldots, n-1, \ j = 0, \ldots, n-1 \) of the matrix \( \tilde{A} \) can be calculated from

\[ \tilde{A}_{ij} = \frac{2}{\pi} \sum_{k=0}^{n-1} c_k g_{ijk}, \]

where \( g_{ijk} \) is given by

\[ g_{ijk} = \int_0^\infty R_i(x)R_j(x)R_k(x)w(x) \, dx. \]

For example, for \( n = 5 \), \( \tilde{A} \) has the following form
and for $n = 9$ we have

$$
\tilde{A} = \begin{bmatrix}
c_0 & c_1 & c_2 & c_3 & c_4 \\
c_1 & c_0 + c_2 & c_1 + c_3 & c_2 + c_4 & c_3 + c_5 \\
c_2 & c_1 + c_3 & c_0 + c_2 + c_4 & c_1 + c_3 + c_5 & c_2 + c_4 + c_6 \\
c_3 & c_2 + c_4 & c_1 + c_3 + c_5 & c_0 + c_2 + c_4 + c_6 & c_1 + c_3 + c_5 + c_7 \\
c_4 & c_3 + c_5 & c_2 + c_4 + c_6 & c_1 + c_3 + c_5 + c_7 & c_0 + c_2 + c_4 + c_6 + c_8 \\
c_5 & c_4 + c_6 & c_3 + c_5 + c_7 & c_2 + c_4 + c_6 + c_8 & c_1 + c_3 + c_5 + c_7 \\
c_6 & c_5 + c_7 & c_4 + c_6 + c_8 & c_3 + c_5 + c_7 & c_2 + c_4 + c_6 + c_8 \\
c_7 & c_6 + c_8 & c_5 + c_7 & c_4 + c_6 + c_8 & c_3 + c_5 + c_7 \\
c_8 & c_7 & c_6 + c_8 & c_5 + c_7 & c_4 + c_6 + c_8 \\
\end{bmatrix}
$$
SOLVING VOLTERRA’S POPULATION MODEL

Converting Volterra’s population model to a nonlinear ODE

In this section, we convert Volterra’s population model in Eq. (1) to an equivalent nonlinear ordinary differential equation. Let

$$y(x) = \int_0^x u(t)dt,$$  (12)

this leads to

$$y'(x) = u(x), y''(x) = u'(x).$$  (13)

Inserting Eqs. (12) and (13) into Eq. (1) yields the nonlinear differential equation

$$ky'''(x) = y'(x) - (y'(x))^2 - y(x)y'(x),$$  (14)

with the initial conditions

$$y(0) = 0, \quad y'(0) = u_0.$$  (15)

Applying RSC tau method on Volterra’s population model

To solve Eq. (14) with initial conditions in Eq. (15), we approximate $y(x)$ and $y^{(j)}(x), j = 1, 2$, using RSC functions as

$$y(x) \approx \sum_{i=0}^{n-1} c_i R_i(x) = A^T R(x),$$  (16)

(notice, for simplicity, that we assume $c_i^R = c_i$)

$$y^{(j)}(x) \approx \sum_{i=0}^{n-1} c_i R_i^{(j)}(x) = A^T D^j R(x), \quad j = 1, 2,$$  (17)

where $D^j$ is the $j$th power of the operational matrix of derivative $D$ given in Eq. (8). Using the product operational matrix in Eq. (11) and Eqs. (16)-(17) we get

$$y''(x) \approx A^T DR(x)R^T(x)D^T A = A^T DR(x)R^T(x)F = A^T D\tilde{F}R(x),$$  (18)
and

$$y'(x)y(x) \approx A^T D R(x) R^T(x) A = A^T D \tilde{A} R(x),$$  \hspace{1cm} (19)

where $F = D^T A$. Using Eqs. (17) - (19) we define the residual $Res(x)$ for Eq. (14) as

$$Res(x) = (kA^T D^2 - A^T D + A^T D \tilde{F} + A^T D \tilde{A}) R(x).$$  \hspace{1cm} (20)

As in a typical tau method (Canuto et al., 1986) we generate $(n - 2)$ algebraic equations by applying

$$< Res(x), R_k(x) >= \int_0^\infty Res(x) R_k(x) w(x) dx = 0, \hspace{0.5cm} k = 0, 1, \ldots, n - 3. \hspace{1cm} (21)$$

In addition, from Eqs. (15) - (17) two additional algebraic equations are obtained as

$$y(0) = A^T R(0) = 0, \hspace{0.5cm} y'(0) = A^T D R(0) = u_0.$$  

Solving this system of $n$ algebraic equations from unknown coefficients of the vector $A$ in Eq. (16), the approximate solutions can be obtained.

**ILLUSTRATIVE EXAMPLES**

**The Volterra’s population model**

We applied the method presented in this paper to examine the mathematical structure of $u(t)$ in the Volterra’s model. In particular, we seek to study the rapid growth along the logistic curve that will reach a peak, then followed by the slow exponential decay where $u(t) \to 0$ as $t \to \infty$. The mathematical behavior was introduced by Scudo (1971) and justified by Small & Klamkin (1989) by using singular perturbation methods for the inner and outer solutions. Further, these properties were also confirmed in TeBeest (1997) by using a phase plane analysis and in Wazwaz (1999) by using Padé approximation. We applied the method presented in this paper and solved Eq. (14) for $u_0 = 0.1$ and $k = 0.02, 0.04, 0.1, 0.2,$ and $0.5$ with $n = 5, 9$ and then evaluated $u_{\text{max}}$ in $t_{\text{critical}}$, which are also evaluated in Wazwaz (1999). In Table 1, the resulting values of $u_{\text{max}}$ using the present method for $n = 5$ and $n = 9$ together with the results obtained in Wazwaz (1999) and the exact values are presented. Figures 1 and 2 show the graphs of $u(t)$ obtained using the present method for $u_0 = 0.1$ and different values of $k$ and $n$. 
Table 1. Comparison between the method in Wazwaz (1999), the present method for $n = 5, 9$ and the exact values for $u_{\text{max}}$

<table>
<thead>
<tr>
<th>$k$</th>
<th>$\text{Present n = 5}$</th>
<th>$\text{Wazwaz (1999)}$</th>
<th>$\text{Exact u}_{\text{max}}$</th>
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<td>Critical $t$</td>
<td>$u_{\text{max}}$</td>
<td>Critical $t$</td>
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<tr>
<td>0.02</td>
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<td>0.04</td>
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<td>0.1</td>
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<td>0.2</td>
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<td>0.680613</td>
<td>0.736620</td>
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<tr>
<td>0.5</td>
<td>0.574684</td>
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</table>

Fig. 1. Graphs of $u(t)$ for $u_0 = 0.1$, $n = 5$ and different values of $k$.

The Lane-Emden equation of index $m$

To solve Eq. (2) using RSC tau method, we first multiply Eq. (2) by $\frac{1}{x+1}$ to get

$$\frac{x}{x+1}y''(x) + \frac{2}{x+1}y'(x) + \frac{x}{x+1}y^m(x) = 0.$$  \hspace{1cm} (22)
Expressing the coefficient functions in Eq. (22) in terms of RSC functions implies

\[
\frac{x}{x + 1} = \sum_{i=0}^{n-1} e_i R_i(x) = \begin{bmatrix}
\frac{1}{2} & & & \\
\frac{1}{4} & 0 & & \\
0 & \ddots & \ddots & \\
& \ddots & \ddots & 0
\end{bmatrix} R(x) = E^T R(x),
\]

(23)

\[
\frac{2}{x + 1} = \sum_{i=0}^{n-1} f_i R_i(x) = \begin{bmatrix}
1 & & & \\
\frac{-1}{2} & 0 & & \\
& \ddots & \ddots & 0
\end{bmatrix} R(x) = F^T R(x).
\]

(24)

From Eq. (16) we get

\[
y^2(x) \approx A^T R(x) R^T(x) A = A^T \tilde{A} R(x),
\]

and therefore

\[
y^m(x) \approx A^T \tilde{A}^{m-1} R(x), m \in \mathbb{N}.
\]

(25)

**Fig. 2.** Graphs of \(u(t)\) for \(u_0 = 0.1\), \(n = 9\) and different values of \(k\).

Using Eq. (17) and Eqs. (23) - (25) we obtain

\[
\frac{x}{x + 1} y''(x) \approx A^T D^2 R(x) R^T(x) E = A^T D^2 \tilde{E} R(x),
\]

(26)
\[
\frac{2}{x+1}y'(x) \approx A^T DR(x)R^T(x)F = A^T D\tilde{F}R(x), \quad (27)
\]
\[
\frac{x}{x+1}y^m(x) \approx A^T \tilde{A}^{m-1} R(x)R^T(x)E = A^T \tilde{A}^{m-1} \tilde{E}R(x). \quad (28)
\]

The matrices $\tilde{E}$ and $\tilde{F}$ can be calculated similarly to Eq. (11). From Eqs. (26) - (28) the residual $Res(x)$ for Eq. (22) can be written as

\[
Res(x) = [A^T D^2 \tilde{E} + A^T D\tilde{F} + A^T \tilde{A}^{m-1} \tilde{E}] R(x).
\]

Substituting the above residual in Eq. (21), we generate $(n-2)$ algebraic equations. Moreover, from Eqs. (3), (16) and (17) we generate two additional algebraic equations as

\[
y(0) = A^T R(0) = 1, \quad y'(0) = A^T DR(0) = 0.
\]

![Fig. 3. Lane-Emden equation graphs for $n = 4$ and $m = 3, 4$.](image)

Consequently, the solutions for Lane-Emden equation of index $m$ can be calculated. Figs. 3 and 4 show the solutions of Lane-Emden equation by the present method with $m = 3, 4$ for $n = 4$ and $n = 9$, respectively, which are in full agreement with Zwillinger (1997) and Weisstein (1999).
CONCLUSION

The operational matrices of derivative and product of RSC functions together with the tau method have been utilized for solving the Volterra’s model for population growth of a species within a closed system and the Lane-Emden equation as nonlinear initial value problems over infinite intervals. The stability and convergence of the second kind Chebyshev approximations make this approach very attractive and contributed to the good agreement between approximate and exact values for the numerical examples.

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تقريب شيبيشيف النسيم من النوع الثاني

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خلاصه

لحل بعض المسائل الفیزيائیة على فترات شبه لا منتهیة. نعطي في هذا البحث تقنية عددية جديدة لحل بعض المسائل الفیزيائیة على فترات شبه لا منتهیة. وتبنى طريقتنا على طريقة شيبيشيف - تاو النسبیة من النوع الثانی. ونقدم كذلك المصفوفات الهرمیة لمشتقه وجد ودال شيبيشيف من النوع الثانی. كما تقوم أيضاً بحل مثالين غير خطیين في المثال الأول نأخذ نموذج فولتيریا للتکاثر ونعتبر عنه على شكل معادلة تفاضلیة غير خطیة. في المثال الثانی ندرس معادلة لاين - إمتد التفاضلیة غير الخطیة. ثم نستعمل طريقتنا لتحويل حلول هذه المسائل الفیزيائیة إلى حلول لأنظمة معادلات جبریة. وتعتبر طريقتنا سهلة الاستخدام وقادرة على الوصول إلى نتائج دقيقة.
الاشتراكات

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