NSE characterization of projective special linear group $L_3(7)$

SHITIAN LIU*

*School of Science, Sichuan University of Science and Engineering, Zigong Sichuan, 643000, P. R. China

ABSTRACT

Let $G$ be a group and $\omega(G)$ be the set of element orders of $G$. Let $k \in \omega(G)$ and $s_k$ be the number of elements of order $k$ in $G$. Let $\text{nse}(G) = \{s_k|k \in \omega(G)\}$. In Khatami et al and Liu's works, the groups $L_3(2)$, $L_3(4)$ and $L_3(5)$ are unique determined by $\text{nse}(G)$. In this paper, we prove that if $G$ is a group such that $\text{nse}(G)=\text{nse}(L_3(7))$, then $G \cong L_3(7)$.

Keywords: Element order; projective special linear group; Thompson’s problem; number of elements of the same order; simple group.

INTRODUCTION

In 1987, J. G. Thompson posed a very interesting problem related to algebraic number fields as follows (Shi, 1989).

Thompson’s Problem. Let $T(G) = \{(n,s_n)|n \in \omega(G) \text{ and } s_n \in \text{nse}(G)\}$, where $s_n$ is the number of elements with order $n$. Suppose that $T(G) = T(H)$. If $G$ is a finite solvable group, is it true that $H$ is also necessarily solvable?

A finite group $G$ is called a simple $K_n$-group, if $G$ is a simple group with $|\pi(G)| = n$.

It was proved that: Let $G$ be a group and $M$ some simple $K_i$-group, $i = 3, 4$, then $G \cong M$ if and only if $| = |M|$ and $\text{nse}(G) = \text{nse}(M)$ (Shao et al., 2009; Shao et al., 2008). And also the group $A_{12}$ is characterizable by order and nse (Liu & Zhang, 2012). Recently, all sporadic simple groups have been proved to be characterizable by nse and order (Asboceli et al., 2013).

Comparing the sizes of elements of same order but disregarding the actual orders of elements in $T(G)$ of the Thompson’s Problem, in other words, it remains only nse $(G)$, whether can it characterize finite simple groups? Up to now, some groups especial for $L_2(q)$, where $q \in \{7, 8, 9, 11, 13\}$, can be characterized by only the set nse $(G)$ (Khatami et al., 2011; Shen et al., 2010). The author has proved that the groups $L_3(4)$ and $L_3(5)$ are characterizable by nse (Liu 2013). In this paper, it is shown that the group $L_3(7)$ also can be characterized by nse.
Here we introduce some notations which will be used. Let $a, b$ denote the product of an integer $a$ by an integer $b$. If $n$ is an integer, then we denote by $\pi(n)$ the set of all prime divisors of $n$. Let $G$ be a group. The set of element orders of $G$ is denoted by $\omega(G)$. Let $k \in \omega(G)$ and $s_k$ be the number of elements of order $k$ in $G$. Let $\text{nse}(G) = \{s_k|k \in \omega(G)\}$. Let $\pi(G)$ denote the set of prime $p$ such that $G$ contains an element of order $p$. $L_n(q)$ denotes the projective special linear group of degree $n$ over finite fields of order $q$. $U_n(q)$ denotes the projective special unitary group of degree $n$ over finite fields of order $q$. The other notations are standard (Conway et al., 1985).

**SOME LEMMAS**

**Lemma 1.** (Frobenius, 1895) Let $G$ be a finite group and $m$ be a positive integer dividing $|G|$. If $L_m(G) = \{g \in G|g^m = 1\}$, then $m || L_m(G)$.

**Lemma 2.** (Miller, 1904) Let $G$ be a finite group and $p \in \pi(G)$ be odd. Suppose that $P$ is a Sylow $p$-subgroup of $G$ and $n = p^m$ with $(p, m) = 1$. If $P$ is not cyclic and $s > 1$, then the number of elements of order $n$ is always a multiple of $p^s$.

**Lemma 3.** (Shen et al., 2010) Let $G$ be a group containing more than two elements. If the maximal number $s$ of elements of the same order in $G$ is finite, then $G$ is finite and $|G| \leq s(s^2 - 1)$.

**Lemma 4.** (Hall, 1959) Let $G$ be a finite solvable group and $|G| = mn$, where $m = p_1^{a_1} \cdots p_r^{a_r}, (m, n) = 1$. Let $\pi = \{p_1, \cdots, p_r\}$ and $h_m$ be the number of Hall $\pi$-subgroups of $G$. Then $h_m = q_1^{b_1} \cdots q_s^{b_s}$ satisfies the following conditions for all $i \in \{1, 2, \cdots, s\}$:

1. $q_i^{b_i} \equiv 1(\text{mod } p_j)$ for some $p_j$.

2. The order of some chief factor of $G$ is divided by $q_i^{b_i}$.

**Lemma 5.** (Shao & Jiang, 2010) Let $G$ be a finite group, $P \in \text{Syl}_p(G)$, where $p \in \pi(G)$. Suppose that $G$ has a normal series $K \triangleleft L \triangleleft G$ and $p || |K|$, then the following statements hold:

1. $N_{G/K}(PK/K) = N_{G}(P)K/K$.

2. If $P \leq L$, then $|G : N_G(P)| = |L : N_L(P)|$, namely, $n_p(G) = n_p(L)$.

3. If $P \leq L$, then $|L/K : N_{L/K}(PK/K)l_t = |G : N_G(P)| = |L : N_L(P)|$, namely, $n_p(L/K)t = n_p(G) = n_p(L)$, for some integer $t$. In particular, $|N_K(P)|t = |K|$

To prove $G \cong L_3(7)$, we need the structure of simple $K_4$-groups.
Lemma 6. (Shi, 1991) Let $G$ be a simple $K_4$-group. Then $G$ is isomorphic to one of the following groups:

1. $A_7, A_8, A_9$ or $A_{10}$.
2. $M_{11}, M_{12}$ or $J_2$.
3. One of the following:
   a. $L_2(r)$, where $r$ is a prime and $r^2 - 1 = 2^a \cdot 3^b \cdot v^c$ with $a \geq 1, b \geq 1, c \geq 1$, and $v$ is a prime greater than $3$.
   b. $L_2(2^m)$, where $2^m - 1 = u, 2^m + 1 = 3^t$ with $m \geq 2, u, t$ are primes, $t > 3, b \geq 1$.
   c. $L_2(3^m)$, where $3^m + 1 = 4t, 3^{m-1} = 2u^c$ or $3^m + 1 = 4t^b, 3^{m-1} = 2u$ with $m \geq 2, u, t$ are odd primes, $b \geq 1, c \geq 1$.
4. One of the following 28 simple groups: $L_2(16), L_2(25), L_2(49), L_2(81), L_3(4), L_3(5), L_3(7), L_3(8), L_3(17), L_4(3), L_4(4), S_4(5), S_4(7), S_4(9), S_6(2), O_8^+(2), G_2(3), U_3(4), U_3(5), U_3(7), U_3(8), U_3(9), U_4(3), U_5(2), Sz(8), Sz(32),^2 D_4(2)$ or $2F_4(2)'$.

Lemma 7. Let $G$ be a simple $K_4$-group and $\{19\} \subseteq \pi(G) \subseteq \{2, 3, 7, 19\}$. Then $G \cong L_3(7)$.

Proof. From Lemma 6(1)(2), order consideration rules out this case.

So we consider Lemma 6(3). We will deal with this with the following cases. 

Case 1. $G \cong L_2(r)$, where $r \in \{3, 7, 19\}$.

Let $r = 3$. Then $|\pi(r^2 - 1)| = 1$, which contradicts $|\pi(r^2 - 1)| = 3$.

Let $r = 7$, then $|\pi(r^2 - 1)| = 2$, which contradicts $|\pi(r^2 - 1)| = 3$.

Let $r = 19$, then $|\pi(r^2 - 1)| = 3$. Hence $G \cong L_2(19)$, but $5 \mid |G|$, a contradiction.

Case 2. $G \cong L_2(2^m)$, where $u \in \{3, 7, 19\}$.

Let $u = 3$, then $m = 2$ and so $5 = 3t^b$. But the equation has no solution in $N$, a contradiction.

Let $u = 7$, then $m = 3$, and $2^3 + 1 = 3t^b$. Thus $t = 3$ and $b = 1$. But $t > 3$, a contradiction.

Let $u = 19$, then $2^m - 1 = 19$. But the equation has no solution in $N$.

Case 3. $G \cong L_2(3^m)$.

We will consider this case by the following two cases.

Subcase 3.1. $3^m + 1 = 4t$ and $3^m - 1 = 2u^c$. 
We can suppose that \( t \in \{3, 7, 19\} \).

Let \( t = 3, 19 \), the equation \( 3^m + 1 = 4t \) has no solution. So we rule out the case.

Let \( t = 7 \), then \( m = 3 \) and so \( 3^3 - 1 = 211 \), which means \( 11 | |G| \), a contradiction.

Subcase 3.2. \( 3^m + 1 = 4^p 3^m + 1 = 4^p \) and \( 3^m - 1 = 2u \).

We can suppose that \( u \in \{3, 7, 19\} \).

Let \( u = 3, 7, 19 \), then the equation \( 3^m - 1 = 2u \) has no solution in \( \mathbb{N} \), a contradiction.

In review of Lemma 6(4), order consideration, \( G \cong L_3(7) \).

This completes the proof of the Lemma.

**MAIN THEOREM AND ITS PROOF**

Let \( G \) be a group such that \( nse (G) = nse (L_3(7)) \), and \( s_n \) be the number of elements of order \( n \) By Lemma 3, we have that \( G \) is finite. We note that \( s_n = k \varphi(n) \), where \( k \) is the number of cyclic subgroups of order \( n \). Also we note that if \( n > 2 \), then \( \varphi(n) \) is even. If \( m \in \omega(G) \), then by Lemma 1 and the above discussion, we have

\[
\varphi(m) | s_m \\
\frac{m}{\sum_{d|m} s_d}\quad(1)
\]

**Theorem 1.** Let \( G \) be a group with \( nse (G) = nse (L_3(7)) = 1,2793, 52136, 117306, 117648, 134064, 156408, 234612, 469224, \ 59270 \), where \( L_3(7) \) is the projective special linear group of degree 3 over finite field of order 7. Then \( G \cong L_3(7) \).

**Proof.** We prove the theorem by first proving that \( \pi(G) \subseteq \{2, 3, 7, 19\} \), secondly showing that \( |G| = |L_3(7)| \), and finally conclude that \( G \cong L_3(7) \).

By (1), \( \pi(G)\{2, 3, 5, 7, 17, 19, 117307, 234613\} \) If \( m > 2 \), then \( \varphi(m) \) is even, then \( s_2 = 2793, 2 \in \pi(G) \).

In the following, we prove that \( 17 \notin \pi(G) \). If \( 17 \in \pi(G) \), then by (1), \( s_{17} = 592704 \). If \( 2.17 \in \omega(G) \), then by Lemma 1, \( 2.17 | 1 + s_2 + s_{17} = s^2_{2.17} \notin nse(G) \). Therefore \( 2.17 \notin \omega(G) \). It follows that the Sylow 17-subgroup \( P_{17} \) of \( G \) acts fixed point freely on the set of elements of order 2 and \( |P_{17}| | s_2 \), a contradiction. Similarly by (1), we can prove that the primes \( 117307, 234613 \notin \pi(G) \).
Hence we have $\pi(G)\{2, 3, 5, 7, 19\}$. Furthermore, by (1) $s_3 = 52136, s_5 = 134064, 469224$ or $592704, s_7 = 117648$ and $s_{19} = 592704$.

If $2^a \in \omega(G)$, then $\varphi(2^a) = 2^{a-1} s_{2^a}$ and so $0 \leq a \leq 7$.

By Lemma 1, $|P_2| = 1 + s_2 + s_2^2 + \ldots + s_2^7$ and so $|P_2| | 2^7$.

If $3^a \in \omega(G)$, then $1 \leq a \leq 4$.

Let exp $(P_3) = 3$. Then by Lemma 1, $|P_3| = 1 + s_3$ and $|P_3| | 3$.

Let exp $(P_3) = 3^2$. Then by Lemma 1, $|P_3| = 1 + s_3 + s_3^2$ and $|P_3| | 3^2$ (when $s_9 = 592704$).

Let exp $(P_3) = 3^3$. Then by Lemma 1, $|P_3| = 1 + s_3 + s_3^2 + s_3^3$ and $|P_3| | 3^6$ (when $s_9 = 117306, s_{27} = 134064$).

Let exp $(P_3) = 3^4$. Then by Lemma 1, $|P_3| = 1 + s_3 + s_3^2 + s_3^3 + s_3^4$ and $|P_3| | 3^4 \land 4$ (when $s_9 = 117648, s_{27} = 469224$ and $s_{81} = 592704$).

Therefore $|P_3| | 3^6$.

If $2^2.3 \in \omega(G)$, then by (Shao & Jiang, 2014), $s_{22.3} = 2.s_3.t$ for some integer $t$. But the equation has no solution since $s_{22.3} \in nse(G)$. Therefore $2^2.3 \not\in \omega(G)$. Similarly $2.3^3 \not\in \omega(G)$.

If $5^a \in \omega(G)$, then $a = 1$.

If $2.5 \in \omega(G)$, then by (Shao & Jiang, 2014), $s_{2.5} = s_5.t$ for some integer $t$ and so $s_{2.5} = s_5$. But by Lemma 1, $2.5 | 1 + s_2 + s_5 + s_{2.5} (270922, 941242, 1188202)$, a contradiction. Therefore $2.5 \not\in \omega(G)$. Similarly $3.5 \not\in \omega(G)$.

If $7^a \in \omega(G)$, then $1 \leq a \leq 4$.

Let exp $(P_7) = 7$. Then by Lemma 1, $|P_7| = 1 + s_7$ and $|P_7| | 7^6$.

Let exp $(P_7) = 7^2$. Then by Lemma 1, $|P_7| = 1 + s_7 + s_7^2$ and $|P_7| | 7^3$ (when $s_{72} \{117306, 156408, 234612, 469224, 592704\}$).

Let exp $(P_7) = 7^3$. Then by Lemma 1, $|P_7| = 1 + s_7 + s_7^2 + s_7^3$ and $|P_7| | 7^3$ (when $s_{72} \in \{117306, 156408, 234612, 469224, 592704\}$).

Let exp $(P_7) = 7^4$. Then by Lemma 1, $|P_7| = 1 + s_7 + s_7^2 + s_7^3 + s_7^4$ and $|P_7| | 7^4$.

Therefore $|P_7| | 7^6$.

If $3.7 \in \omega(G)$, then by (Shao & Jiang 2014), $s_{3.7} = 2.s_7.t$ for some integer $t$. But the equation has no solution since $s_{3.7} \in nse(G)$. Therefore $3.7 \not\in \omega(G)$. Similarly $4.7 \not\in \omega(G)$.

If $19^a \in \omega(G)$, then $1 \leq a \leq 2$. Since $s_{19} \not\in nse(G)$, then $a = 1$. By Lemma 1, $|P_{19}| = 1 + s_{19}$ and $|P_{19}| | 19$. 
If $2.19 \in \omega(G)$, then by (Shao & Jiang 2014), $s_{2.19} = s_{1.9}$. By Lemma 1, $2.19 \mid 1 + s_2 + s_{1.9} + s_{2.19}$, a contradiction. Hence $2.19 \notin \omega(G)$. Similarly $3.19, 5.19, 7.19 \notin \omega(G)$.

To remove the prime 5, we assume that $7 \in \pi(G)$.

If $3, 5, 19 \notin \pi(G)$. Then $G$ is a 2-group. Since $\omega(G) = 8$ and there are exactly ten numbers in nse ($L_3(7)$) and consequently, we have a contradiction.

Let $19 \in \pi(G)$. Then since $|P_{19}| = 19, n_{19} = s_{1.9}/\varphi(19) = 2^5 . 3^7.3^3$ and $7 \in \pi(G)$, a contradiction.

Let $5 \in \pi(G)$. We know that $s_5 = 134064, 469224, 592704$.

Let $s_5 = 134064$. Then by Lemma 1, $|P_5| \mid 1 + s_5$ and $|P_5| = 5$ Since $n_5 = s_5/\varphi(5) = 2^2 . 3^2 . 7^2 . 19$, a contradiction.

Let $s_5 = 469224$. Then $|P_5| = 5^2$.

If $|P_5| = 5$, then $n_5 = 2^3 . 3^7 . 19, 7, 19 \in \pi(G)$, a contradiction.

If $|P_5| = 5^2$, then we can assume that $\{2, 3, 5\} \subset \pi(G)$. Therefore $1876896 + 52136k_1 + 117306k_2 + 117648k_3 + 134064k_4 + 156408k_5 + 234612k_6 + 469224k_7 + 592704k_8$ $= 2^a . 3^b . 5^2$ where $k_1, k_2, ..., k_8a, b$ are non-negative integers and $0 \leq \sum_{k=0}^{8} k_i \leq 2$, then the equation has no solution in N.

Let $s_5 = 592704$. Then $|P_5| = 5$. Since $n_5 = s_5/\varphi(5) = 2^4 . 3^7 . 3^7, 7 \in \pi(G)$, a contradiction.

Let $3 \in \pi(G)$.

Let $\exp(P_3) = 3$. Then $|P_3| \mid 3^3$.

If $|P_3| = 3$, then since $n_3 = s_3/\varphi(3) = 2^2 . 3^7 . 19, 7, 19 \in \pi(G)$, a contradiction.

If $|P_3| = 3^2$, then $1876896 + 52136k_1 + 117306k_2 + 117648k_3 + 134064k_4 + 156408k_5 + 234612k_6 + 469224k_7 + 592704k_8 = 2^a . 3^2$ where $k_1, k_2, ..., k_8, a$ are non-negative integers and $0 \leq \sum_{k=0}^{8} k_i \leq 0$, then the equation has no solution in N.

Similarly we can rule out the other case $|P_3| = 3^3$.

Let $\exp(P_3) = 3^2$. Then $|P_3| \mid 3^4$ (when $s_9 = 592704$).

If $|P_3| = 3^2$, then $7 \in \pi(G)$, a contradiction.

If $|P_3| = 3^3$, then $1876896 + 52136k_1 + 117306k_2 + 117648k_3 + 134064k_4 + 156408k_5 + 234612k_6 + 469224k_7 + 592704k_8 = 2^a . 3^3$, where $k_1, k_2, ..., k_8, a$ are non-negative integers and $0 \leq \sum_{k=0}^{8} k_i \leq 1$, then the equation has no solution in N.
Similarly we can rule out the case $|P_3| = 3^4$.

Let $\exp (P_3) = 3^3$. Then $|P_3| \mid 3^6$ (when $s_9 = 117306, s_{27} = 134064$).

If $|P_3| = 3^3$. Then since $n_3 = s_{33}/\varphi(3^3)$, 7 or 19 $\in \pi(G)$, a contradiction.

If $|P_3| > 3^3$, we also can rule out.

Let $\exp (P_3) = 3^4$.

If $|P_3| = 3^4$, then since $s_{81} = 592704), n_3 = s_{34}/\varphi(3^4) = 2^5.7^3,7 \in \pi(G)$, a contradiction.

If $|P_3| > 3^4$, then by Lemma 2, $s_{34} = 3^4t$ for some integer $t$. But the equation has no solution since $s_{34} \in nse(G)$

Therefore $7 \in \pi(G)$.

If $5.7 \in \omega(G)$, then by Lemma 1, $5.7 \mid 1 + s_5 + s_7 + s_{5.7}$. But $s_{5.7} \notin nse(G)$. So $5.7 \notin \omega(G)$. It follows that the Sylow 5-subgroup of $G$ acts fixed point freely on the set of elements of order 7 and $|P_5| \mid s_7$, a contradiction. So $5 \notin \pi(G)$.

If $19 \in \pi(G)$, then since $n_{19} = s_{19}/\varphi(19) = 2^5.3.7^3,3,7 \in \pi(G)$, then we only have to consider two proper sets $\{2,7\}, \{2,3,7\}$ and finally the whole set $\{2,3,7,19\}$.

Case a. $\pi(G) = \{2,7\}$.

We know that $\exp (P_7) = 7, 7^2, 7^3, 7^4$.

Let $\exp (P_7) = 7$. Then $|P_7| \mid 1 + s_7$ and so $|P_7| \mid 7^6$.

Let $|P_7| = 7$. Then since $n_7 = s_7/\varphi(7) = 117648/6 = 2^3.3.19.43,19 \in \pi(G)$, a contradiction.

Let $|P_7| = 7^2$. Then $1876896 + 52136k_1 + 117306k_2 + 117648k_3 + 134064k_4 + 156408k_5 + 234612k_6 + 469224k_7 + 592704k_8 = 2^a.49$, where $k_1, k_2, ..., k_8, a$ are non-negative integers and $0 \leq \sum_{k=0}^8 k_i \leq 0$. So the equation has no solution in N.

If $|P_7| > 7^2$, similarly we can rule out as the case “$\exp(P_7) = 7$ and $|P_7| = 7^2$“.

Let $\exp (P_7) = 7^2$. Then $|P_7| \mid 7^3$.

Let $|P_7| = 7^2$. Then $s_{72} \in \{117306, 134064, 156408, 234612, 469224, 592704\}$. Since $n_7 = s_{72}/\varphi(7^2), 3$ or 19 $\in \pi(G)$.

If $|P_7| = 7^3$ Then $1876896 + 52136k_1 + 117306k_2 + 117648k_3 + 134064k_4 + 156408k_5 + 234612k_6 + 469224k_7 + 592704k_8 = 2^a.7^3$
where $k_1, k_2, \ldots, k_8, a$, are non-negative integers and $0 \leq \sum_{k=0}^{8} k_i \leq 1$. So the equation has no solution in $\mathbb{N}$.

Let $\exp (P_7) = 7^3$. Then $|P_7| = 7^3$. Since $s_{73} \in \{117306, 134064, 156408, 234612, 469224, 592704\}$ and $n_7 = s_{73}/\varphi(7^3)$, 3 or 19 $\in \pi(G)$.

Let $\exp (P_7) = 7^4$. Then $|P_7| = 7^4$. Since $s_{74} \in \{117306, 156408, 234612, 469224, 592704\}$ and $n_7 = s_{73}/\varphi(7^3)$, 3 or 19 $\in \pi(G)$.

Case b. $\pi(G) = \{2, 3, 7\}$.

Let $\exp (P_7) = 7$. Then $|P_7| \mid 1 + s_7$ and so $|P_7| \mid 7^6$.

Let $|P_7| = 7$. Then since $n_7 = s_7/\varphi(7) = 117648/6 = 2^3 \cdot 3 \cdot 19 \cdot 43, 19 \in \pi(G)$, a contradiction.

Let $|P_7| = 7^2$. Then $1876896 + 52136k_1 + 117306k_2 + 117648k_3 + 134064k_4 + 156408k_5 + 234612k_6 + 469224k_7 + 592704k_8 = 2^a \cdot 3^b \cdot 49$ where $k_1, k_2, \ldots, k_8, a, b$ are non-negative integers and $0 \leq \sum_{k=0}^{8} k_i \leq 5$. Since $1876896 \leq |G| = 2^a \cdot 3^b \cdot 72 \leq 1876896 + 5.92704$, the equation has no solution in $\mathbb{N}$.

If $|P_7| > 7^2$, similarly we can rule out as above.

Let $\exp (P_7) = 7^2$. Then $|P_7| \mid 7^3$ and $1876896 + 52136k_1 + 117306k_2 + 117648k_3 + 134064k_4 + 156408k_5 + 234612k_6 + 469224k_7 + 592704k_8 = 2^a \cdot 3^b \cdot 49$ where $k_1, k_2, \ldots, k_8, a, b$ are non-negative integers and $0 \leq \sum_{k=0}^{8} k_i \leq 6$. Since $1876896 \leq |G| = 2^a \cdot 3^b \cdot 72 \leq 1876896 + 6.92704$ It follows that the number of Sylow 2-subgroups of $G$ is 1, 3, 7, 9, 15, 21, 27, 49, 63, 81, 147, 189, 243, 441, 189, 243, 441, 567, 729, 1323, 1701, 3969, 5103, 11907, 35721 and so the number of elements of order 2 is 1, 3, 7, 9, 15, 21, 27, 49, 63, 81, 147, 189, 243, 441, 189, 243, 441, 567, 729, 1323, 1701, 3969, 5103, 11907, 35721, but none of which belongs to nsc $(G)$.

Let $\exp (P_7) = 7^3$ Then $|P_7| \mid 7^3$ and $1876896 + 52136k_1 + 117306k_2 + 117648k_3 + 134064k_4 + 156408k_5 + 234612k_6 + 469224k_7 + 592704k_8 = 2^a \cdot 3^b \cdot 49$ where $k_1, k_2, \ldots, k_8, a, b$ are non-negative integers and $0 \leq \sum_{k=0}^{8} k_i \leq 8$. We can rule out this case as “$\exp(P_7) = 7^2$”.

Let $\exp (P_7) = 7^4$ Then $|P_7| \mid 7^4$ and $1876896 + 52136k_1 + 117306k_2 + 117648k_3 + 134064k_4 + 156408k_5 + 234612k_6 + 469224k_7 + 592704k_8 = 2^a \cdot 3^b \cdot 74$ where $k_1, k_2, \ldots, k_8, a, b$ are non-negative integers and $0 \leq \sum_{k=0}^{8} k_i \leq 9$. We can rule out this case as “$\exp(P_7) = 7^2$”.
Case c. $\pi(G) = \{2, 3, 7, 19\}$.

In the following, we first show that $|G| = 2^5.3^2.7^3.19$, or $|G| = 2^6.3^2.7^3.19$, secondly prove that $G \cong L_3(7)$.

Step 1. $|G| = 2^5.3^2.7^3.19$, or $|G| = 2^6.3^2.7^3.19$.

From the above arguments, we have that $|P_{19}| = 19$.

We know $7.19 \notin \omega(G)$. It follows that the Sylow 7-subgroup of $G$ acts fixed freely on the set of elements of order 19 and so $|P_7| \mid s_{19}$. Therefore $|P_7| \mid 7^3$. Similarly $2.19 \notin \omega(G)$ and $|P_2| \mid 2^6; 3.7 \notin \omega(G)$ and $|P_3| \mid 3^2$.

Therefore we have $|G| = 2^m.3^n.7^p.19$. But $1876896 = 2^5.3^2.7^3.19 \leq 2^m.3^n.7^p.19$. Therefore $|G| = 2^5.3^2.7^3.19$, $|G| = 2^5.3^3.7^3.19$, $|G| = 2^6.3^3.7^3.19$, or $|G| = 2^6.3^3.7^3.19$.

Step 2. $G \cong L_3(7)$

First show that there is no group such that $|G| = 2^6.3^3.7^3.19$, and nse($G$) = nse($L_3(7)$). Then get the result by (Shao et al., 2008).

Let $|G| = 2^6.3^2.7^3.19$ and nse ($G$) = nse($L_3(7)$).

Let $G$ be soluble. Since $s_{19} = 592704$, then $n_{19} = s_{19}/\phi(19) = 2^5.3.7^3$. By Lemma 4, $3 \equiv 1$ (mod 19), a contradiction. So $G$ is insoluble.

Therefore we can suppose that $G$ has a normal series $1 \leq K \leq L \leq G$ such that $L/K$ is isomorphic to a simple $K_{11}$-group with $i= 3, 4$ as $19^2 \nmid |G|$.

If $L/K$ is isomorphic to a simple $K_3$ -group, then from (Herzog, 1968), $L/K \cong L_2(7), L_2(8)$.

From (Conway et al., 1985), $n_7(L/K) = n_7(L_2(7)) = 8$, and so $n_7(G) = 8t$ for some integer $t$ and $7 \times t$. Hence the number of elements of order 7 in $G^{\pi_7}; s_7 = 8t.6 = 48t$. Since $s_7 \in \text{nse}(G)$, then $s_7 = 117648$ and so $t = 2451$. Therefore $3.19.43 \mid |K| \mid 2^2.3.7^2.19$, which is a contradiction. For $L_2(8)$, similarly we can rule out.

If $L/K$ is isomorphic to a $K_4$-group, then by Lemma 7, $L/K \cong L_3(7)$.

Let $\bar{G} = G/K$ and $\bar{L} = L/K$. Then

$L_3(7) \leq \bar{L} \cong \bar{L}C_{\bar{G}}(\bar{L})/C_{\bar{G}}(\bar{L}) \leq \bar{G}/C_{\bar{G}}(\bar{L}) = N_{\bar{G}}(\bar{L})/C_{\bar{G}}(\bar{L}) \leq \text{Aut}(\bar{L})$

Set $M = \{xK | xK \in C_{\bar{G}}(\bar{L})\}$, then $G/M \cong \bar{G}/C_{\bar{G}}(\bar{L})$ and so $L_3(7) \leq G/M \leq \text{Aut} L_3(7)$

Therefore $G/M \cong L_3(7), G/M \cong 2.L_3(7), G/M \cong 3.L_3(7)$, or $G/M \cong S_3.L_3(7)$.
If $G/M \cong L_3(7)$, then order consideration $|M| = 2$. It follows that $M$ is a normal subgroup generated by a 2-central element of $G$. So there exists an element of order $2.19$, which is a contradiction.

If $G/M \cong 3.L_3(7)$, or $G/M \cong S_3.L_3(7)$, order consideration rules out these cases.

If $G/M \cong 2.L_3(7)$, then $M = 1$. But $\text{nse}(2.L_3(7)) \neq \text{nse}(G)$.

Therefore $|G| = 2^5.3^2.7^3.19 = |L_3(7)|$ and by assumption, $\text{nse}(G) = \text{nse}(L_3(7))$, then by (Shao et al., 2008), $G \cong L_3(7)$.

This completes the proof of the theorem.

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توصيف الزمر الاسقاطية الخطية الخاصة

شيتان ليو
كلية العلوم - جامعة سيتشوان للعلوم والهندسة - زيجوج سيتشوان
643000 - جمهورية الصين الشعبية

خلاصة

ثبت في هذا البحث بأنه إذا W, G مجموعة عناصر مرتبتة لنكن زمرة وتكون G تكافيئ (L₃(7)) فإن nse(G) = nse(L₃(7)).