Some applications of differential subordination on certain class of analytic functions defined by integral operator

R. M. EL-ASHWAH* AND M. K. AOUF**

* Department of Mathematics., Faculty of Science, Damietta University, Egypt
** Department of Mathematics., Faculty of Science, Mansoura University, Egypt

ABSTRACT

Two parameters function $H(n, \lambda; z)$ involving the Feltt multiplier operator is introduced. Subordination properties as well as sufficient conditions for starlikeness are also obtained.

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INTRODUCTION

Let $A$ denote the class of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

which are analytic in the open unit disc $U = \{z \in C : \ |z| < 1\}$. Let $S$, $S^*(\alpha)$ and $C(\alpha)$ $(0 \leq \alpha < 1)$ be the subclasses of functions in $A$ which are, respectively, univalent, starlike of order $\alpha$ and convex of order $\alpha$ in $U$. Denote by $S^*(0) = S^*$ and $C(0) = C$. Suppose also that $P$ denotes the class of functions $k(z)$ given by

$$k(z) = 1 + \sum_{k=1}^{\infty} c_k z^k,$$

which are analytic in $U$ and satisfy the inequality

$$Re(k(z)) > 0 \quad (z \in U).$$

If $f$ and $g$ are analytic in $U$, we say that $f$ is subordinate to $g$, written $f(z) \prec g(z)$ if there exists a Schwarz function $w(z)$, which (by definition) is analytic in $U$ with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in U$, such that $f(z) = g(w(z))$, $z \in U$. Furthermore, if the function $g$ is univalent in $U$, then we have the following equivalence (Miller & Mocanu, 2000; Bulboacă, 2005):
\[ f(z) \prec g(z) \iff f(0) = g(0) \text{ and } f(U) \subset g(U). \]

For a function \( f(z) \) given by (1.1) and \( g(z) \) defined by

\[ g(z) = z + \sum_{k=2}^{\infty} b_k z^k, \quad (1.3) \]

the Hadamard product (or convolution) of \( f(z) \) and \( g(z) \) is defined by

\[ (f \ast g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g \ast f)(z). \quad (1.4) \]

For an analytic function \( f(z) \) given by (1.1) and for \( n \in \mathbb{N}_0 \), Flett (1972) defined the multiplier transformations \( P^n f \) by

\[ P^n f(z) = z + \sum_{k=2}^{\infty} k^{-n} a_k z^k \quad (z \in U; \ n \in \mathbb{N}_0). \quad (1.5) \]

Clearly, the function \( P^n f(z) \) is analytic in \( U \). We note that for \( n \in \mathbb{N}_0 \), we have

(i) \[ P^{-n} f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k = D^n f(z), \]

and

(ii) \[ z( P^{-n} f(z))' = D^{n+1} f(z), \]

where the operator \( D^n f \) was introduced by Sălăgean (1983). We also note that

\[ P^n( P^m f(z)) = P^{n+m} f(z) \quad (z \in U) \]

for all integers \( n \) and \( m \). Further, the operator \( P^n \) can be seen as a convolution of two functions. That is

\[ P^n f(z) = (h \ast h \ast \ldots \ast h \ast f)(z), \]

where the function \( h(z) = \log \frac{1}{1-z} = z + \sum_{k=2}^{\infty} k^{-1} z^k \) occurs \( n \) times. It follows from (1.5) that

\[ z(P^n f(z))' = P^{-1} f(z) \quad (n \in \mathbb{Z}) \quad (1.6) \]

and

\[ P^0 f(z) = f(z), \quad P^{-1} f(z) = z f'(z), \quad P^{-2} f(z) = z(f(z) + z f'(z)). \]

We now define a two-parameters function \( H(n, \lambda; z) \) by
\[ H(n, \lambda; z) = (1 - \lambda) \frac{f^{(n-1)}(z)}{f^{(n)}(z)} + \lambda \frac{f^{(n-2)}(z)}{f^{(n-1)}(z)} \quad (z \in U; \lambda \in R; n \in Z; f \in A). \quad (1.7) \]

Finally, we denote by \( K(n, \lambda, \alpha) \) the class of functions \( f(z) \in A \), which satisfy the following condition:

\[ \text{Re}(H(n, \lambda; z)) > \alpha \quad (z \in U; 0 \leq \alpha < 1; \lambda \in R; n \in Z). \]

We note that:

(i) \( K(0, \lambda, \alpha) = M(\lambda, \alpha) (\lambda \geq 0; 0 \leq \alpha < 1) \), is the class of \( \lambda \)-convex functions of order \( \alpha \) (Srivastava & Attiya, 2007);

(ii) \( K(0, \lambda, 0) = M(\lambda) (\lambda \geq 0) \), is the class of \( \lambda \)-convex functions (Miller et al., 1973; Mocanu, 1969; Mocanu, 1994);

(iii) \( K(0, 0, \alpha) = S^*(\alpha) \) and \( K(0, 1, \alpha) = C(\alpha) \) (Srivastava & Owa, 1992).

Consider the first-order differential subordination

\[ H(\varphi(z), z\varphi'(z); z) \prec h(z). \]

A univalent function \( q \) is called its dominant if \( \varphi(z) \prec q(z) \) for all analytic functions \( \varphi \) that satisfy this differential subordination. A dominant \( \hat{q} \) is called the best dominant, if \( q(z) \prec \hat{q}(z) \) for all the dominants \( q \). For the general theory of the first-order differential subordination and its applications, we refer the reader to Bulboacă (2005) and Miller & Mocanu (2000).

**DIFFERENTIAL SUBORDINATION ASSOCIATED WITH** \( H(n, \lambda; z) \)

To establish our main results we shall require the following lemma.

**Lemma 1** (Miller & Mocanu, 1985 and Miller & Mocanu, 2000). Let the function \( q(z) \) be univalent in \( U \), and let the functions \( \theta \) and \( \varphi \) be analytic in a domain \( D \) containing \( q(U) \), with \( \varphi(w) \neq 0 \) when \( w \in q(U) \). Set

\[ Q(z) = zq'(z)\varphi(q(z)) \quad \text{and} \quad h(z) = \theta(q(z)) + Q(z) \]

and suppose that

(i) \( Q(z) \) is a starlike function in \( U \),

(ii) \( \text{Re}\left(\frac{zh'(z)}{Q(z)}\right) > 0 \quad (z \in U). \)

If \( p \) is analytic in \( U \) and \( p(0) = q(0), p(U) \subseteq D \) and

\[ \theta(p(z)) +zp'(z)\varphi(p(z)) \prec \theta(q(z)) + zq'(z)\varphi(q(z)), \quad (2.1) \]
then \( p(z) \prec q(z) \), and \( q \) is the best dominant of (2.1).

**Theorem 1.** Let \( \lambda \in \mathbb{R} \setminus \{0\} \), \( n \in \mathbb{Z} \) and \( f(z) \in A \). Suppose also that the function \( q(z) \) univalent in \( U \), with \( q(0) = 1 \) and \( q(z) \neq 0 \) \((z \in U)\), and satisfies each of the following inequalities:

\[
\text{Re} \left( 1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} \right) > 0 \quad (z \in U)
\]

and

\[
\text{Re} \left( 1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} + \frac{1}{\lambda} q(z) \right) > 0 \quad (z \in U).
\]

If

\[
H(n, \lambda; z) \prec q(z) + \lambda \frac{zq'(z)}{q(z)},
\]

then

\[
\frac{P^{-1}f(z)}{Pf(z)} \prec q(z)
\]

and \( q(z) \) is the best dominant of (2.4).

**Proof.** We choose

\[
g(z) = \frac{P^{-1}f(z)}{Pf(z)}, \quad \theta(w) = w \text{ and } \varphi(w) = \frac{\lambda}{w}.
\]

Then \( \theta(w) \) and \( \varphi(w) \) are analytic in the domain \( C^* = C \setminus \{0\} \), which contains \( q(U) \), \( q(0) = 1 \), and \( \varphi(w) \neq 0 \) when \( w \in q(U) \). Next, we define the functions \( Q(z) \) and \( h(z) \) by

\[
Q(z) = zq'(z)\varphi(q(z)) = \lambda \frac{zq'(z)}{q(z)}
\]

and

\[
h(z) = \theta(q(z)) + Q(z) = q(z) + \lambda \frac{zq'(z)}{q(z)}.
\]

It follows from (2.2) and (2.3) that \( Q(z) \) is starlike in \( U \) and
\[
\Re \left( \frac{zh'(z)}{Q(z)} \right) > 0 \quad (z \in U).
\]

We note also that the function \( g(z) \) is analytic in \( U \), with \( g(0) = q(0) = 1 \), since \( 0 \notin q(U) \). Therefore \( g(U) \subset C^* \). Thus, the hypotheses of Lemma 1 are satisfied and we find that, if

\[
\theta(g(z)) + zg'(z)\varphi(g(z)) = H(n, \lambda; z) \prec h(z),
\]

then

\[
\frac{P^{-1}f(z)}{P^nf(z)} \prec q(z).
\]

and \( q(z) \) is the best dominants.

**Remark 1.** If \( q(z) \in P \) and \( \lambda > 0 \), then we can omit the condition (2.3) in Theorem 1.

**Remark 2.** If \( q(z) \in P \) and \( \lambda < 0 \), then we can omit the condition (2.2) in Theorem 1.

**Theorem 2.** For \( \lambda > 0, n \in \mathbb{Z} \) and \( 0 \leq \alpha < 1 \), if \( f(z) \in A \) and

\[
H(n, \lambda; z) \prec \frac{(1 - 2\alpha)^2 z^2 + 2[(1 - 2\alpha) + \lambda(1 - \alpha)]z + 1}{(1 - z)[1 + (1 - 2\alpha)z]},
\]

then the operator \( Pf(z) \) is a starlike function of order \( \alpha \) in \( U \), that is,

\[
\Re \frac{P^{-1}f(z)}{P^nf(z)} > \alpha \quad (z \in U).
\]

**Proof.** For \( 0 \leq \alpha < 1 \) and \( z \in U \), we first put

\[
q(z) = \frac{1 + (1 - 2\alpha)z}{1 - z} \in P
\]

in Theorem 1. Then, since

\[
\Re \left( 1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} \right) = \Re \left( \frac{1}{1 - z} + \frac{1}{1 + (1 - 2\alpha)z} - 1 \right) > 0 \quad (z \in U),
\]

the proof of Theorem 2 is completed.
Remark 3. Letting $\lambda \to 0^+$ in (2.8), we have

$$\frac{P^{n-1}f(z)}{P^nf(z)} \leq \frac{1 + (1 - 2\alpha)z}{1 - z},$$

which implies that the operator $P^nf(z)$ is a starlike function of order $\alpha$ in $U$, that is, $Re\left\{\frac{P^{n-1}f(z)}{P^nf(z)}\right\} > \alpha \quad (0 \leq \alpha < 1, \ z \in U)$.

By taking $\alpha = 0$ in Theorem 2, we obtain the following result.

**Corollary 1.** For $\lambda > 0$ and $n \in Z$, if $f(z) \in A$ and

$$H(n, \lambda; \ z) \prec \frac{z^2 + 2(1 + \lambda)z + 1}{1 - z^2},$$

(2.9)

then the operator $P^nf(z)$ is a starlike function in $U$, that is,

$$Re\left\{\frac{P^{n-1}f(z)}{P^nf(z)}\right\} > 0 \quad (z \in U).$$

**Theorem 3.** Let $\lambda < 0$, $0 \leq \alpha < 1$, $n \in Z$, such that

$$A(\lambda, \alpha; \ z) Re(z) + B(\lambda, \alpha; \ z) > 0 \quad (z \in U),$$

where

$$A(\lambda, \alpha; \ z) = -\lambda(1 - 2\alpha)|1 - z|^2 + [\lambda + 2(1 - \alpha)]|1 + (1 - 2\alpha)z|^2,$$

(2.10)

and

$$B(\lambda, \alpha; \ z) = |1 - z|^2[(1 + \lambda - 2\alpha)|1 + (1 - 2\alpha)z|^2 - \lambda] - [\lambda + 2(1 - \alpha)]|1 + (1 - 2\alpha)z|^2.$$

(2.11)

If $f(z) \in A$ and $H(n, \lambda; \ z)$ satisfies the subordination (2.8), then the operator $P^nf(z)$ is a starlike function of order $\alpha$ in $U$, that is,

$$Re\left\{\frac{P^{n-1}f(z)}{P^nf(z)}\right\} > \alpha \quad (z \in U; \ 0 \leq \alpha < 1).$$

**Proof.** For $0 \leq \alpha < 1$, $\lambda < 0$ and $z \in U$, we first set $q(z) = \frac{1 + (1 - 2\alpha)z}{1 - z}$ in Theorem 1, to obtain,
\[ 1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} + \frac{1}{\lambda} q(z) = \frac{1 + [\lambda + (1 - 2\alpha)]z}{\lambda(1 - z)} + \frac{1}{1 + (1 - 2\alpha)z}. \]

Then, after some calculations, we observe that

\[ \text{Re}\left( 1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} + \frac{1}{\lambda} q(z) \right) = -\frac{A(\lambda, \alpha; z)\text{Re}(z) + B(\lambda, \alpha; z)}{\lambda|1 - z|^2|1 + (1 - 2\alpha)z|^2} > 0 \quad (z \in U), \]

which completes the proof of Theorem 3.

By taking \( \alpha = 0 \) in Theorem 3, we obtain the following result.

**Corollary 2.** For \( \lambda < 0 \), such that

\[ (\lambda + 1)(1 + |z|^2) + 2\text{Re}(z) < 0 \quad (z \in U). \tag{2.12} \]

If \( f(z) \in A \) and the function \( H(n, \lambda; z) \) satisfies the subordination relation (2.9), then the operator \( I^n f(z) \) is starlike in \( U \).

**Proof.** For \( \lambda < 0 \) and \( z \in U \), we first set \( q(z) = \frac{1 + z}{1 - z} \), in Theorem 1, to obtain,

\[ 1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} + \frac{1}{\lambda} q(z) = \frac{1 + (\lambda + 1)z}{\lambda(1 - z)} + \frac{1}{1 + z}. \]

Then, after some calculations, we observe that

\[ \text{Re}\left( 1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} + \frac{1}{\lambda} q(z) \right) = \frac{(1 - |z|^2)((\lambda + 1)(1 + |z|^2) + 2\text{Re}(z))}{\lambda|1 - z|^2|1 + z|^2} > 0 \quad (z \in U), \]

which completes the proof of Corollary 2.

Finally, by taking \( \alpha = 0 \) and \( \lambda \leq -2 \) in Theorem 3, we obtain the following sufficient condition for starlikeness.

**Corollary 3.** For \( \lambda \leq -2 \) and \( z \in U \), if the function \( H(n, \lambda; z) \) satisfies the subordination relation (2.9), then the operator \( f(z) \) is starlike in \( U \).

**Proof.** Since \( \text{Re}(z) \leq |z| < 1, \ z \in U \), then from (2.12) for \( \lambda \leq -2 \), we have

\[ (\lambda + 1)(1 + |z|^2) + 2\text{Re}(z) < 0 \quad (z \in U), \]

which proves Corollary 3.
REFERENCES


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بعض تطبيقات التبعية التفاضلية على صنف خاص من الدوال التحليلية
المعروفة بمؤثر تكاملي

رابحة الأشوح و **محمد كمال عوف

*قسم الرياضيات - كلية العلوم - جامعة دمياط
**قسم الرياضيات - كلية العلوم - جامعة المنصورة

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أ. د. سعاد عبد الوهاب العبد الرحمه

P.o.Box: 26585 - Safat. 13126 kuwait
Tel: (+965) 4817689 - 4815453 Fax: (+965) 4812514
E-mail: ajh@kuniv.edu.kw http://www.puhcouncil.kuniv.edu.kw/ajh/