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Some asymptotic stability properties of extreme order statistics under power normalization

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ABSTRACT

In this paper the continuation of the restricted convergence of the power normalized extreme order statistics, on the half-line of real numbers to the whole-line, is proved under general conditions. Moreover, the continuation property of the power normalized extremes, with random sample indices, is proved in an important practical case.

Keywords: Continuation of the convergence; P-max stable laws; Power normalization; Sample of random size; Weak convergence.

INTRODUCTION

Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables (r.v.'s) with common distribution function (d.f.) $F(x) = P(X_n < x)$ and $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n}$ be the corresponding order statistics (o.s). Suppose there exists a nondegenerate d.f. $\Phi_1(x)$ and some suitable norming constants $a_n > 0$ and b_n such that

$$F^n(a_n x + b_n) \xrightarrow{w} \Phi_1(x), \quad \text{as} \quad n \longrightarrow \infty, \tag{1.1}$$

where ($\xrightarrow{w}$) denotes the weak convergence. We call $\Phi_1(x)$ a max stable d.f. under linear normalization or simply ℓ-max stable d.f. It is well known that (see Galambos 1987) every ℓ-max stable d.f. $\Phi_1(x)$ can only have the form (in which x may be replaced by $ax + b$ for any $a > 0, b$) $\Phi_1(x) = \exp(-\mathcal{U}_{i;\alpha}(x))$, $i = 1, 2, 3,\ \alpha > 0$, where

$$\text{Types I):} \qquad \mathcal{U}_{1;\alpha}(x) = \begin{cases} \infty, & x \leq 0, \\ x^{-\alpha}, & x > 0; \end{cases}$$

$$\text{Types II):} \qquad \mathcal{U}_{2;\alpha}(x) = \begin{cases} (-x)^{\alpha}, & x \leq 0, \\ 0, & x > 0; \end{cases}$$

$$\text{Type III):} \qquad \mathcal{U}_3(x) = \mathcal{U}_{3;\alpha}(x) = e^{-x}, \ \forall\ x. \tag{1.2}$$

If Eq. (1.1) holds, we write $F \in D_\ell(\Phi_1)$. Moreover, Eq. (1.1) holds with $\Phi_1(x) = \exp(-\mathcal{U}_{i;\alpha}(x))$, if and only if

$$n(1 - F(a_n x + b_n)) \longrightarrow \mathcal{U}_{i;\alpha}(x). \tag{1.3}$$

Finally, the d.f. of the normalized term $\frac{1}{a_n}(X_{n-r+1:n} - b_n)$ converges weakly to a nondegenerate d.f. $\Phi_r(x) = 1 - \Gamma_r(\mathcal{U}_{i;\alpha}(x))$, if and only if Eq. (1.3) holds, where $\Gamma_r(.)$ is an incomplete gamma function (see Galambos 1987). Following Pantcheva (1985) and Mohan & Ravi (1992) a d.f. F is said to belong to the max domain of attraction of a d.f. $\Psi_1(x)$ under power normalization if there exist norming constant $a_n > 0$ and $b_n > 0$, such that, as $n \to \infty$,

$$P\left(\left|\frac{X_{n:n}}{a_n}\right|^{\frac{1}{b_n}} \mathcal{S}(X_{n:n}) < x\right) = F^n(a_n |x|^{b_n}\ \mathcal{S}(x)) \xrightarrow{w} \Psi_1(x), \tag{1.4}$$

where $\mathcal{S}(x) = -1$ if $x < 0$, $= 0$ if $x = 0$ and $= 1$ if $x > 0$. In this case we write $F \in D_p(\Psi_1)$. We call $\Psi_1(x)$ a max stable d.f. under power normalization or simply p-max stable d.f. if Eq. (1.4) holds. We say that two d.f.'s F_1 and F_2 are of the same p-type if there exist $A > 0$ and $B > 0$ such that $F_1(x) = F_2(A\,|x|^B\,\mathcal{S}(x))\ \forall\, x$. Pantcheva (1985) showed that any p-max stable d.f. $\Psi_1(x)$ can be a p-type of one of the six d.f.'s $\exp(-\mathcal{V}_{i;\beta}(x)),\ i = 1, 2, \ldots, 6,\ \beta > 0$, where

Types I): $$\mathcal{V}_{1;\beta}(x) = \begin{cases} \infty, & x \le 1, \\ (\log x)^{-\beta}, & x > 1; \end{cases}$$

Types II): $$\mathcal{V}_{2;\beta}(x) = \begin{cases} \infty, & x \le 0, \\ (-\log x)^{\beta}, & 0 < x \le 1, \\ 0, & x > 1; \end{cases}$$

Types III): $$\mathcal{V}_{3;\beta}(x) = \begin{cases} \infty, & x \le -1, \\ (-\log(-x))^{-\beta}, & -1 < x \le 0, \\ 0, & x > 0; \end{cases}$$

Types IV): $$\mathcal{V}_{4;\beta}(x) = \begin{cases} (\log(-x))^{\beta}, & x \le -1, \\ 0, & x > -1; \end{cases}$$

Type V): $\mathcal{V}_{5;\beta}(x) = \mathcal{V}_5(x) = \mathcal{U}_{1;1}(x)$;

Type VI): $\mathcal{V}_{6;\beta}(x) = \mathcal{V}_6(x) = \mathcal{U}_{2;1}(x)$ (1.5)

The necessary and sufficient conditions for a d.f. to belong to $D_p(.)$ for each of the six p-max stable laws is obtained by Mohan & Ravi (1992) and Subramanya (1994). In these papers the known results concerning linear normalization were extended to p-stable laws. They showed that every d.f. attracted to ℓ-max stable law is necessarily attracted to some p-max stable and that p-max stable laws, in fact, attract more. Therefore the advantage of normalizing with power functions is given by the fact that p-stable laws attract more d.f.'s than linear-stable laws, which means that the normalization with power functions improves the accuracy of the approximation of the distribution of $X_{n:n}$ for large values of n. A unified approach to the results of Mohan & Ravi (1992) and Subramanya (1994) has been obtained by Christoph & Falk (1996). It is not difficlt to obtain the following slight generalization of the above results for the extreme o.s under power normalization.

Theorem 1.1. (see Barakat and Nigm 2002). For suitable normalizing constants $a_n > 0$ and $b_n > 0$, the d.f. of the normalized term $\left|\frac{X_{n-r+1:n}}{a_n}\right|^{\frac{1}{b_n}} \mathcal{S}(X_{n-r+1:n})$ converges weakly to a nondegenerate d.f. $\Psi_r(x)$ if and only if

$$n\Big(1 - F(a_n|x|^{b_n}\,\mathcal{S}(x))\Big) \longrightarrow \mathcal{V}_{i;\beta}(x), \quad i \in \{1, 2, \ldots, 6\}. \tag{1.6}$$

Moreover, $\Psi_r(x) = 1 - \Gamma_r(\mathcal{V}_{i;\beta}(x))$.

The continuation property of extreme order statistics under linear normalization has been studied by many authors (see, e.g., Gnedenko & Gnedenko 1982 and Gnedenko & Senusi Bereksi 1982 a,b; 1983) in the following two assertions:

(1) If the d.f. of the suitably linear normalized maximum (minimum) o.s converges to a nondecreasing function $\Psi(x)$, for all continuity points of $\Psi(x)$, in a restricted set (interval) $\mathcal{S} = (c, d)$, where $c < 0$, $d > 0$, $\Psi(d) - \Psi(c) > 0$ and $\Psi(x)$ is equal to one of the extreme value distributions for all $x \in (c, d)$, then this convergence will continue weakly, for all x, to this maximum (minimum) value distribution.

(2) If the d.f. of the suitably linear normalized extreme o.s converges to a nondecreasing function $\Psi(x)$, for all continuity points of $\Psi(x)$, in a set $\mathcal{S} = \{x : x \leq A\}$, A is a constant (half-line), where $\Psi(x)$ has at least two growth points on $\mathcal{S}$ and $\Psi(-\infty) = 0$, then the convergence will continue weakly, for all x, to an extreme value distribution which coincides with $\Psi(x)$ on $(-\infty, A]$.

Gnedenko & Sherif (1983) and Barakat (1997) discussed the first assertion for extreme o.s. under linear normalization. Gnedenko *et al.* (1985) investigated the second assertion for the joint d.f. of the rth and sth $(r < s)$ extremes, with the same original d.f. $F(x)$. Recently, in Barakat (2000), the continuation property of the extreme o.s (under linear normalization) is investigated, when $\mathcal{S}$ is considered as a countable set of real values (set of measure zero). Many authors considered the problem of extremes (under linear normalization) with random indices in two different cases. The first case is: when the basic variables $X_1, X_2, \ldots, X_n$ and the random sample size ν_n are independent and the d.f. of $\frac{\nu_n}{n}$ converges weakly to a nondegenerate d.f (see Gnedenko and Gnedenko 1982, Barakat 1990, 1998 and Barakat & Nigm 1999). The second case is: when the interrelation of the basic variables $X_1, X_2, \ldots, X_n$ and the random sample size ν_n is not restricted and $\frac{\nu_n}{n}$ converges in probability $(\xrightarrow{p})$ to a positive r.v. τ (see, for example, Galambos 1978, 1987, Barakat & El-Shandidy 1990, Barakat, 1998 and Barakat & Nigm 1999). Recently, Barakat & Nigm (2002), obtained the sufficient conditions for the convergence of the extremes with random sample size ν_n, under power normalization, as well as the limit forms of the d.f.'s, where the normalizing constants do not contain the random size ν_n (stability of the power normalizing constants). In this paper two theorems concerning some stability properties of the weak convergence of the extreme o.s under power normalization are proved. The first theorem (Theorem 2.1) proved that the convergence of the sequence of the d.f.'s of the suitably power normalized extremes on the interval $(-\infty, A]$, $A < \infty$, to an arbitrary nondecreasing function will continue on the whole line to p-extremes types. The second theorem (Theorem 3.1) deals with the aforesaid continuation property (given in the second assertion) for the maximum o.s under power normalization, when the sample size itself is a random variable ν_n.

CONTINUATION OF CONVERGENCE FOR THE *P*-EXTREME TYPES

Before formulating the main result in this section, we introduce two concepts.

Definition 2.1. Let F_n be a sequence of d.f.'s Then the restricted convergence $F_n(x) \xrightarrow{\mathcal{S}} F(x)$, as $n \to \infty$, where $\mathcal{S}$ is a set of real numbers and F is a nondecreasing and nonnegative function, means that the convergence of F_n to the limit F is restricted on $\mathcal{S}$, for all continuity points of F.

Definition 2.2. Let $\mathcal{S}$ be a set of real numbers. Then a function $F(x)$ is said to be nondegenerate on $\mathcal{S}$ if it has at least two growth points on $\mathcal{S}$.

Theorem 2.1. Let A be a real number and $a_n > 0$ and $b_n > 0$, be suitable power normalizing constants, for which

$$\Psi_{r:n}\Big(a_n\ |x|^{b_n}\ \mathcal{S}(x)\Big) = P\left(\left|\frac{X_{n-r+1:n}}{a_n}\right|^{\frac{1}{b_n}}\ \mathcal{S}(X_{n-r+1:n}\right) \overset{\{x\leq A\}}{\longrightarrow} \Psi_{or}(x), \qquad (2.1)$$

where $\Psi_{or}(x)$ is nondegenerate function on the interval $x \leq A$. Then the sequence $\Psi_{r:n}\Big(a_n\ |x|^{b_n}\ \mathcal{S}(x)\Big)$ converges, for all x, to a nondegenerate d.f. $\Psi_r(x)$, where $\Psi_r(x) = \Psi_{or}(x),\ \forall\, x \leq A$. Moreover, $\Psi_r(x) = 1 - \Gamma_r(\mathcal{V}_i(x)), i \in \{1, 2, \ldots, 6\}$.

Since the Proof of Theorem 2.1 is somewhat lengthy, we split it into several Lemmas, some of which are of independent interest.

Lemma 2.1. For any real number A, fixed integer r and suitable power normalizing constants $a_n > 0$ and $b_n > 0$, $\Psi_{r:n}\Big(a_n\ |x|^{b_n}\ \mathcal{S}(x)\Big) \overset{\{x\leq A\}}{\longrightarrow} \Psi_{or}(x)$, if and only if $U_n\Big(a_n\ |x|^{b_n}\ \mathcal{S}(x)\Big) = n\Big(1 - F\Big(a_n\ |x|^{b_n}\ \mathcal{S}(x)\Big)\Big) \overset{\{x\leq A\}}{\longrightarrow} \mathcal{V}_o(x)$, where $\Psi_{or}(x)$ is a nondecreasing and nondegenerate function on the interval $x \leq A$ and $\mathcal{V}_o(x)$ is nonincreasing and nonnegative function, which is determined by the equation $\Psi_{or}(x) = 1 - \Gamma_r(\mathcal{V}_o(x)), x \leq A$.

Proof. This lemma is a special case of Theorem 2.2.1 in Leadbetter *et al.* (1983) (by putting $u_n = a_n\ |x|^{b_n}\ \mathcal{S}(x), \forall\, x \leq A$, and $\tau = \Psi_{or}, \forall\, x \leq A$.

Lemma 2.2. Let $c, d, d > c$ be two real numbers and $a_n > 0$ and $b_n > 0$, be suitable power normalizing constants. Assume F_n is a sequence of d.f.'s such that, as $n \to \infty$, $F_n(x) \overset{[c,d]}{\longrightarrow} F_1(x)$, where $F_1(x)$ is a nondegenerate function on the interval $x \leq A$. furthermore, let $F_n\Big(a_n\ |x|^{b_n}\ \mathcal{S}(x)\Big) \overset{[c,d]}{\longrightarrow} F_2 9x)$, where $F_2(x)$ is a nondegenerate function on the interval $[c, d]$. Finally, let $\{n_k\}$ be an arbitrary sequence of index such that, as $n \to \infty$, $F_{n_k}(x) \overset{w}{\longrightarrow} F_3(x)$, where $F_3(x)$ is a nondegenerate d.f. Then, there exist constants $a > 0$ and $b > 0$ such that $F_2(x) = F_3\Big(a\ |x|^b\ \mathcal{S}(x)\Big), \forall\, x \in [c, d]$.

Proof. The proof is similar to the proof of Lemma 2.4 in Barakat (1997).

Remark 2.1. Clearly, the statement of Lemma 2.2 is valid if the interval $[c, d]$ is replaced by the interval $\{x : x \leq A\}$.

Lemma 2.3. Under the assumptions of Theorem 2.1, the sequence $\Big\{\psi_{r:n}\Big(a_n\ |x|^{b_n}\ \mathcal{S}(x)\Big)\Big\}$ is stochastically bounded (for definition see Barakat 1997).

Proof. It suffices to show that, for any subsequence $\{n_k\}$ for which

$$\Psi_{r:n_k}\left(a_{n_k}\ |x|^{b_{n_k}}\ \mathcal{S}(x)\right) \stackrel{w}{\longrightarrow} \tilde{\Psi}_r(x), \tag{2.2}$$

$\tilde{\Psi}_r(\infty) = 1$. Let us assume the converse, i.e.,

$$\tilde{\Psi}_r(\infty) < 1. \tag{2.3}$$

On the other hand, in view of Lemma 2.1, Eq. (2.2) is equivalent to $U_{n_k}\left(a_{n_k}\ |x|^{b_{n_k}}\ \mathcal{S}(x)\right) \longrightarrow \tilde{\mathcal{V}}(x)$, where $\tilde{\Psi}_r(x) = 1 - \Gamma_r(\tilde{V}(x))$. Consequently, Eq. (2.3) is equivalent to

$$\tilde{\mathcal{V}}(\infty) > 0. \tag{2.3$'$}$$

Furthermore, in view of Eq. (2.1) and Lemma 2.1, for all integers m we get

$$mU_n\left(a_{nm}\ |x|^{b_{nm}}\ \mathcal{S}(x)\right) = U_{nm}\left(a_{nm}\ |x|^{b_{nm}}\ \mathcal{S}(x)\right) \stackrel{x\le A}{\longrightarrow} \mathcal{V}_o(x), \quad \text{as } n \to \infty,$$

where $\Psi_{or}(x) = 1 - \Gamma_r(\mathcal{V}_o(x))$, $x \le A$.
Thus, as $n \to \infty$, we get $U_n\left(a_{nm}\ |x|^{b_{nm}}\ \mathcal{S}(x)\right) \stackrel{x\le A}{\longrightarrow} \frac{\mathcal{V}_o(x)}{m} = \mathcal{V}^\star(x)$, which, in view of Lemma 2.1, yields

$$\Psi_{r:n}\left(a_{nm}\ |x|^{b_{n}m}\ \mathcal{S}(x)\right) \stackrel{x\le A}{\longrightarrow} \Psi_r^\star(x), \tag{2.4}$$

where $\Psi_r^\star(x) = 1 - \Gamma_r(\mathcal{V}^\star(x))$. Applying now Lemma 2.2 to Eq. (2.1), Eq. (2.2) and Eq. (2.4), we can find $\alpha_m > 0$ and $\beta_m > 0$, such that $\tilde{\Psi}_r\left(\alpha_m\ |x|^{\beta_m}\ \mathcal{S}(x)\right) = \Psi_r^\star(x)$, $x \le A$. This in view of Lemma 2.1, yields $\tilde{\mathcal{V}}\left(\alpha_m\ |x|^{\beta_m}\ \mathcal{S}(x)\right) = \mathcal{V}^\star(x) = \frac{\mathcal{V}_o(x)}{m}$, $x \le A$. Since the two functions $\tilde{\mathcal{V}}$ and $\mathcal{V}_o$ are non increasing, then $\mathcal{V}_o(A) \ge m\tilde{\mathcal{V}}\left(\alpha_m\ |x|^{\beta_m}\ \mathcal{S}(x)\right) \ge m\tilde{\mathcal{V}}(\infty)$. Hence, by virtue of assumption Eq. (2.3)$'$ and by letting $m \to \infty$, we get $\mathcal{V}_o(A) = \infty$, which in turn yields $\Psi_{or}(A) = 0$. This contradicts the fact that Ψ_{or} is nondegenerate on $x \le A$, which completes the proof.

Before turning to the proof of Theorem 2.1 we first consider the following remark.

Remark 2.2. Under the conditions of Lemma 2.3, if there exists an integer $m > 1$ such that $\alpha_m\ |A|^{\beta_m}\ \mathcal{S}(A) \le A$, then

$$\tilde{\Psi}_r\left(\alpha_m\ |A|^{\beta_m}\ \mathcal{S}(A)\right) = \Psi_{or}\left(m\ |A|^{\beta_m}\ \mathcal{S}(A)\right) \le \Psi_{or}(A).$$

In view of Lemma 2.1, this leads to $\frac{\mathcal{V}_o(A)}{m} = \tilde{\mathcal{V}}\Big(\alpha_m\ |A|^{\beta_m}\ \mathcal{S}(A)\Big) \geq \mathcal{V}_o(A)$. This in turn leads to $\mathcal{V}_o(A) = 0$ or ∞. but the last value ∞ is impossible because this implies $\Psi_{or}(A) = 0$. Hence, $\mathcal{V}_o(A) = 0$, i.e., $\Psi_{or}(A) = 1$. this means that the convergence wil be continued in this case for all x. This completes the proof.

Proof of Theorem 2.1. Let us assume $A < \alpha_m\ |A|^{\beta_m}\ \mathcal{S}(A)$, $\forall\, m > 1$. Under this assumption, if the convergence in Eq. (2.1) is proved to be continued for all x, then, in view of Remark 2.2, Theorem 2.1 will immediately follow. On the other hand, if $\mathcal{S}(A) = 0$, then $\alpha_m\ |A|^{\beta_m}\ \mathcal{S}(A) \leq A$ is trivially satisfied. Therefore, again in view of Remark 2.2, henceforth, we assume $A \neq 0$. Furthermore, let us consider the following three cases:

A) There exists an integer $m > 1$ such that $\beta_m < 1$.

B) There exists an integer $m > 1$ such that $\beta_m = 1$.

C) For all integers $m > 1$, we have $\beta_m > 1$.

Case A): Clearly, in view of the assumption $A < \alpha_m\ |A|^{\beta_m}\ \mathcal{S}(A)$, we have $A < \alpha_m^{\frac{1}{1-\beta_m}}$. We shall show that in this case the convergence in Eq.(2.1) will be continued to the point $B = \alpha_m^{\frac{1}{1-\beta_m}}\mathcal{S}(A)$. On the other hand, by using Remark 2.2, the convergence will be continued for all x (since, $\alpha_m\ |B|^{\beta_m}\ \mathcal{S}(B) = \alpha_m B^{\beta_m}\mathcal{S}(A) = \alpha_m^{\frac{1}{1-\beta_m}}\mathcal{S}(A) = B$). Indeed, we have, as $n \to \infty$

$$\Psi_{r:n}\Big(a_{nm}\ |x|^{b_nm}\ \mathcal{S}(x)\Big) \overset{x \leq A}{\longrightarrow} \Psi_r^{\star}(x) = \tilde{\Psi}_r\Big(\alpha_m\ |x|^{\beta_m}\ \mathcal{S}(x)\Big). \tag{2.5}$$

Putting $y = \alpha_m\ |x|^{\beta_m}\ \mathcal{S}(x)$ in Eq.(2.5), we get

$$\Psi_{r:n}\left(a_{nm}\left|\frac{y}{\alpha_m}\right|^{\frac{b_{nm}}{\beta_m}}\mathcal{S}(y)\right) = \Psi_{r:n}\Big(a_{1nm}\ |y|^{b_{1nm}}\ \mathcal{S}(y)\Big) \overset{y \leq A_1}{\longrightarrow} \tilde{\Psi}_r(y), \text{ as } n \to \infty, \tag{2.6}$$

where $a_{1nm} = \alpha_m^{-\frac{b_{nm}}{\beta_m}}$, $b_{1nm} = \frac{b_{nm}}{\beta_m}$ and $A_1 = \alpha_m\ |A|^{\beta_m}\ \mathcal{S}(A)$. By application of Lemma 2.2 (Remark 2.1) to Eq.(2.1) and Eq.(2.6), we get (note that $\forall\, x \leq A < \alpha_m\ |A|^{\beta_m}\ \mathcal{S}(A)$, $\tilde{\Psi}_r(y) \equiv \Psi_{or}(x)$)

$$\left(\frac{a_n}{a_{1nm}}\right)^{\frac{1}{b_{1nm}}} = \frac{\alpha_m}{a_{nm}} a_n^{\frac{\beta_m}{b_{nm}}} \to 1 \quad \text{and} \quad \frac{b_n}{b_{1nm}} = \frac{\beta_m b_n}{b_{nm}} \to 1, \quad \text{as } n \to \infty.$$

By a further application of Lemma 2.2 (Remark 2.2), the two sequences a_{1nm} and b_{1nm} in Eq.(2.6) can be changed to a_n and b_n, respectively. Thus, we get

$$\Psi + r : n\left(a_n \ |y|^{b_n} \ \mathcal{S}(y)\right) \overset{y \leq A_1}{\longrightarrow} \tilde{\Psi}_r(y), \text{ as } n \to \infty.$$

Repeating this argument N times yields the relation

$$\Psi_{r:n}\left(a_n \ |y|^{b_n} \ \mathcal{S}(y)\right) \overset{y \leq A_N}{\longrightarrow} \tilde{\Psi}_r(y), \text{ as } n \to \infty,$$

where

$$A_N = \alpha_m \ |A_{N-1}|^{\beta_m} \ \mathcal{S}(A_{N-1}) = \ldots = \alpha_m^{\frac{1-\beta_m^N}{1-\beta_m}} \ |A|^{\beta_m^n} \ \mathcal{S}(A) \to \alpha_m^{\frac{1}{1-\beta_m}} \ \mathcal{S}(A) = B,$$

as $N \to \infty$. Therefore, due to the continuity of the function $\tilde{\Psi}_r(y)$, the proof of theorem 2.1 follows in this case by virtue of Lemma 2.3.

Case B: In this case, it is easy to show that $\alpha_m \neq 1(\alpha_m < 1,$ if $\mathcal{S}(A) = -1,$ $\alpha_m > 1,$ if $\mathcal{S}(A) = 1)$. Then, starting with Eq.(2.5) (with $\beta_m = 1$) and putting $y = \alpha_m \ |x| \ \mathcal{S}(x)$, we get, as $n \to \infty$,

$$\Psi_{r:n}\left(a_{nm} \ \left|\frac{y}{\alpha_m}\right|^{b_{nm}} \ \mathcal{S}(y)\right) = \Psi_{r:n}\left(a_{2nm} \ |y|^{b_{nm}} \ \mathcal{S}(y)\right) \overset{y \leq A_1'}{\longrightarrow} \tilde{\Psi}_r(y), \text{ as } n \to \infty, \quad (2.7)$$

where $a_{2nm} = a_{nm}\alpha_m^{-b_{nm}}$ and $A_1' = \alpha_m A$. By application of Lemma 2.2 (Remark 2.1) to Eq.(2.1) and Eq.(2.7), we get

$$\left(\frac{a_n}{a_{2nm}}\right)^{\frac{1}{b_{nm}}} = \frac{b_n}{b_n m} \to 1 \text{ and } \frac{b_n}{b_{1nm}} = \frac{\beta_m b_n}{b_{nm}} \to 1, \text{ as } n \to \infty.$$

By a further application of Lemma 2.2 (Remark 2.1), the two sequences a_{2nm} and b_{nm}, in Eq.(2.7) can be changed to a_n and b_n respectively. Hence we get

$$\Psi_{r:n}\left(a_n \ |y|^{b_n} \ \mathcal{S}(y)\right) \overset{y \leq A_N'}{\longrightarrow} \tilde{\Psi}_r(y), \text{ as } n \to \infty,$$

where

$$A_N' = \alpha_m A_{N-1}' = \ldots = \alpha_m^N A \longrightarrow \begin{cases} \infty, & \text{if } \mathcal{S}(A) = 1(\alpha_m > 1), \\ 0, & \text{if } \mathcal{S}(A) = -1(\alpha_m < 1), \end{cases}$$

as $N \to \infty$. The proof of Theorem 2.1 in the case $A'_N \to \infty$ immediately follows, from Lemma 2.3 and the continuity of the function $\tilde{\Psi}_r$ and in the case $A'_N \to 0$, the proof follows on applying Remark 2.1 (since, $\alpha_m\ |A|^{\beta_m}\ \mathcal{S}(A) \leq A$ is trivially satisfied, as we have seen before, when $A = 0$).

Case C: In view of the assumption $A < \alpha_m\ |A|^{\beta_m}\ \mathcal{S}(A)$, we have $\alpha_m\ |A|^{\beta_m - 1} > 1$, if $\mathcal{S}(A) = 1$ and $\alpha_m\ |A|^{\beta_m - 1} < 1$, if $\mathcal{S}(A) = -1$. Therefore, using the same argument in cases A and B, we can prove the continuation of the convergence in Eq.(2.1) for all $x \leq A''_N$, where

$$A''_N = \alpha_m^{\frac{\beta_m^N - 1}{\beta_m - 1}}\ |A|^{\beta_m^N}\ \mathcal{S}(A) = \alpha_m^{\frac{-1}{\beta_m - 1}}\ \left(\alpha_m\ |A|_m^{\beta} - 1\right)^{\frac{\beta_m^n}{\beta_m - 1}}\ \mathcal{S}(A) \to \begin{cases} \infty, & \text{if } \mathcal{S}(A) = 1, \\ 0, & \text{if } \mathcal{S}(A) = -1, \end{cases}$$

as $N \to \infty$. The proof of Theorem 2.1 in the case $A'_N \to \infty$ immediately follows, from Lemma 2.3 and the continuity of the function $\tilde{\Psi}_r$ and in the case $A'_N \to 0$, the proof of the theorem follows on applying Remark 2.1. This completes the proof of the theorem.

THE CONTINUATION OF THE CONVERGENCE FOR THE *P*-MAXIMUM WITH RANDOM SAMPLE SIZES

In this section the sample size itself is considered as a r.v. ν_n, which has a geometric d.f. with mean n. Barakat & Nigm (2002) show that the class of possible nondegenerate limit d.f.'s of the power normalized random maximum o.s $\left|\frac{X_{\nu_n:\nu_n}}{a_n}\right|^{\frac{1}{b_n}}\ \mathcal{S}(X_{\nu_n:\nu_n})$, where $a_n, b_n > 0$ are suitable normalizing constants, contains only the following types of d.f.'s:

$$\mathcal{L}_1^{(1)}(x) = \begin{cases} 0, & x \leq 1, \\ (1nx)^{\beta} & x > 1; \end{cases}$$

$$\mathcal{L}_1^{(2)}(x) = \begin{cases} 0, & x \leq 0, \\ \dfrac{1}{1 + (-1nx)^{\beta}}, & 0 < x \leq 1, \\ 1, & x > 0; \end{cases}$$

$$\mathcal{L}_1^{(3)}(x) = \begin{cases} 0, & x \leq -1, \\ \dfrac{(-1n(-x))^{\beta}}{1 + (-1n(-x))^{\beta}}, & -1 < x \leq 0, \\ 1, & x > 0; \end{cases}$$

$$\mathcal{L}_1^{(4)}(x) = \begin{cases} \dfrac{1}{1+(1n(-x))^{\beta}}, & x \le -1, \\ 1, & x > -1; \end{cases}$$

$$\mathcal{L}_1^{(5)}(x) = \begin{cases} 0, & x \le 0, \\ \dfrac{x}{1+x}, & x > 0; \end{cases}$$

$$\mathcal{L}_1^{(6)}(x) = \begin{cases} \dfrac{1}{1-x}, & x \le 0, \\ 1, & x > 0; \end{cases}$$

The following theorem proves the continuation property of the aforesaid types of d.f.'s.

Theorem 3.1. Let

$$\Psi_{1:\nu_n}\Big(a_n\ |x|^{b_n}\ \mathcal{S}(x)\Big) \stackrel{x \le A}{\longrightarrow} \mathcal{L}_o(x),$$

as $n \to \infty$. Then

$$\Psi_{1:\nu_n}\Big(a_n\ |x|^{b_n}\ \mathcal{S}(x)\Big) \stackrel{w}{\longrightarrow} \mathcal{L}_1^{(i)}(x), \quad \text{as } n \to \infty.$$

Proof. First we shall need the following lemma, which can be proved, by using Lemma 2.2, in analogous fashion to Theorem 2.1 of Barakat & Nigm (2002).

Lemma 3.1. Let $\dfrac{\nu_n}{n} \stackrel{p}{\longrightarrow} \tau$, where τ is a positive r.v. Assume that there are sequences $a_n > 0$ and $b_n > 0$ such that, as $n \to \infty$,

$$U_n\Big(a_n\ |x|^{b_n}\ \mathcal{S}(x)\Big) \stackrel{x \le A}{\longrightarrow} \mathcal{V}_o(x), \quad 0 \le \mathcal{V}_o(x) \le \infty.$$

Then, as $n \to \infty$,

$$P\left(\left|\frac{X_{\nu_n:\nu_n}}{a_n}\right|^{\frac{1}{b_n}} \mathcal{S}(X_{\nu_n:\nu_n}) < x\right) \stackrel{x \le A}{\longrightarrow} \mathcal{L}_o(x) = \int_0^{\infty} \exp(-z\,\mathcal{V}_o(x))dP(\tau < z).$$

We now turn to the proof of Theorem 3.1. Let $\mathcal{L}_o(x) \equiv \mathcal{L}_1^{(1)}(x)$, $\forall\, x \le A$ (say), where A in this case must be greater than one (since $\mathcal{L}_o(x)$ has at least two growth points). Clearly, we can find a subsequence $\{n_k\}$ for which

$U_{n_k}\left(a_{n_k}\ |x|^{b_{n_k}}\ \mathcal{S}(x)\right) \overset{x\leq A}{\longrightarrow} \mathcal{V}(x),\ 0 \leq \mathcal{V}(x) \leq \infty$. Therefore, in view of Lemma 3.1, we get

$$\frac{(\ln x)^{\beta}}{1+(\ln x)^{\beta}} = \int_0^{\infty} e^{-z\mathcal{V}(x)} d(1-e^{-z}) = \int_0^{\infty} \left(\frac{e^{-z\mathcal{V}(x)}}{e}\right)^z dz = \frac{1}{1+\mathcal{V}(x)},\ \forall x \in (-\infty, A],$$

from which we get $\mathcal{V}(x) = (\ln x)^{-\beta}$, $\forall x \in (-\infty, A]$. On the other hand, for any other subsequence $\{n'_k\}$ for which $U_{n'_k}\left(a_{n'_k}\ |x|^{b_{n'_k}}\ \mathcal{S}(x)\right) \overset{x\leq A}{\longrightarrow} \mathcal{V}'(x),\ 0 \leq \mathcal{V}(x) \leq \infty$, we have $\mathcal{V}'(x) = \mathcal{V}(x) = (\ln x)^{-\beta}$, $\forall x \in (-\infty, A]$. Moreover, $\mathcal{V}(x)$ has more than two growth points. Thus, the theorem in this case follows immediately from theorems 2.1 of Barakat & Nigm (2002) and Theorem 3.1. The proofs $\mathcal{L}_o(x) \equiv \mathcal{L}_1^{(i)}(x)$, $i = 2, 3, \ldots, 6$, are similar to the preceding case.

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بعض خواص الاستقرار التقاربية للإحصاءات المرتبة المتطرفة تحت ثوابت اتزان القوى

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خلاصة

يتناول البحث دراسة استمرار التقارب على خط الأعداد الحقيقية للإحصاءات المرتبة المتطرفة ذات الثوابت غير الخطية باعتباره مقيداً على نصف خط الأعداد الحقيقية وذلك تحت شروط عامة لم تدرس من قبل. تم دراسة تلك الخاصية أيضاً للإحصاءات المرتبة المتطرفة ذات الثوابت غير الخطية عندما يكون حجم العينة متغير عشوائي متقطع موجب ومستقل عن مفردات العينة المسحوبة وتلك الحالة لها الكثير من التطبيقات العملية.